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Restricted Range Approximate Solutions of Nonlinear Differential Systems with Boundary Conditions

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This paper develope uniform approximate solutions to systems of nonlinear differential equations with boundary conditions using restricted range approximations. In the process of deriving the approximations an existence theorem is generated for solutions of nonlinear equations. Also, an algorithm is provided for computing these approximations. © 1986 Academic Press, Inc.

1. INTRODUCTION

Consider the boundary value problem

$$\begin{aligned} \mathbf{y}' &= E\mathbf{y} + \mathbf{f}(t, \mathbf{y}), \qquad t \in [0, \tau], \\ M\mathbf{y}(0) &+ N\mathbf{y}(\tau) = \mathbf{b}, \end{aligned} \tag{1.1}$$

where E, M, and N are constant real $n \times n$ matrices such that $E^n = 0$ and the $n \times 2n$ matrix (M, N) has rank n. $\mathbf{b}$ is a constant real $n \times 1$ vector and $\mathbf{f}(t, \mathbf{y})$ is continuous on $[0, \tau] \times R^n$ with values in R^n, where R will denote the set of real numbers. The purpose of this paper is to examine vector polynomial approximations to a solution of (1.1) with respect to a uniform type norm. The paper is mainly intended to extend the work of [3], although many other papers have recently appeared in closely related

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areas. Among those that most influenced this paper are [2, 8] in linear boundary value problems and [1, 7] in nonlinear initial value problems.

The uniform type norms to be used are as follows:

For $\mathbf{f} = (f_1, f_2, ..., f_n)^T$, where $f_i \in C[0, \tau]$, define

$$|\mathbf{f}(t)| = \max_{1 \leqslant i \leqslant n} |f_i(t)|$$

and

$$\|\mathbf{f}\| = \max_{t \in [0, \tau]} |\mathbf{f}(t)|.$$

For scalar functions $\|g\| = \max_{t \in [0, \tau]} |g(t)|$, where $g \in C[0, \tau]$. Also, if $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix, whose components are piecewise continuous on $[0, \tau]$, then define

$$|B(t)| = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |b_{ij}(t)|$$

and

$$\|B\| = \sup_{t \in [0, \tau]} |B(t)|.$$

Throughout this paper we will use the symbol $\mathbf{F}[\mathbf{y}](t)$ to represent $\mathbf{f}(t, \mathbf{y})$. It will also be assumed that the system

$$\begin{aligned} \mathbf{y}' &= E\mathbf{y} \\ M\mathbf{y}(0) + N\mathbf{y}(\tau) &= \mathbf{0} \end{aligned} \tag{1.2}$$

is incompatible. This implies that there exists a unique Green's matrix $G(t, s)$ (see [5]) such that the unique solution $\mathbf{y}$ to the boundary value problem

$$\begin{aligned} \mathbf{x}' &= E\mathbf{x} + \mathbf{g}(t) \\ M\mathbf{x}(0) + N\mathbf{x}(\tau) &= \mathbf{b} \end{aligned} \tag{1.3}$$

can be written as

$$\mathbf{y}(t) = Y(t)\, D^{-1}\mathbf{b} + \int_0^\tau G(t, s)\, \mathbf{g}(s)\, ds,$$

where $Y(t)$ is an $n \times n$ matrix whose columns are n linearly independent solutions of $\mathbf{y}' = E\mathbf{y}$ such that $Y(0) = I$, I is the $n \times n$ identity matrix, and

$$D = MY(0) + NY(\tau).$$

It is well known (see [5]) that the system (1.2) is incompatible if and only if the matrix D is nonsingular.

The ideal situation is to find a sequence $\{\mathbf{p}_k\}$ such that

$$\inf_{\mathbf{p} \in P_k} \|\mathbf{p}' - E\mathbf{p} - \mathbf{F}[\mathbf{p}]\| = \|\mathbf{p}'_k - E\mathbf{p}_k - \mathbf{F}[\mathbf{p}_k]\|, \tag{1.4}$$

where $P_k = \{\mathbf{p}: \mathbf{p} = (p_1, p_2, ..., p_n)^{\mathrm{T}}$, where p_i is a polynomial of degree k or less and $M\mathbf{p}(0) + N\mathbf{p}(\tau) = \mathbf{b}\}$. If $\mathbf{f}(t, \mathbf{y}) = \mathbf{F}[\mathbf{y}](t)$ is linear in $\mathbf{y}$ and (1.1) has a unique solution this sequence can always be found (see [2]). If $\mathbf{F}[\mathbf{y}]$ is not linear in $\mathbf{y}$ such a sequence may not exist or possibly be very difficult to compute. We will use a different approach to finding an appropriate sequence of vector polynomials to approximate a solution of (1.1), although in some cases the method does produce a vector polynomial which satisfies (1.4) for a given k. This method will produce a sequence of polynomials which are fairly easy to compute and provide a good estimation to a solution of (1.1). The method parallels that used in [3, 7] for scalar equations.

2. Approximating Vector Polynomials

Before developing our sequence of vector polynomials some definitions must be presented. Let $\mathbf{h}(t) = Y(t)\, D^{-1}\mathbf{b}$. Since $E^n = 0$, the components of $\mathbf{h}$ are polynomials of degree $n-1$ or less. Also, $\mathbf{h}$ has the property that

$$\begin{aligned} \mathbf{h}' &= E\mathbf{h} \\ M\mathbf{h}(0) + N\mathbf{h}(\tau) &= \mathbf{b}. \end{aligned} \tag{2.1}$$

An important restriction on (1.1) is that there exists a function $\phi(t) \in C[0, \tau]$, with $\phi(t) > 0$ for all $t \in [0, \tau]$, and a positive real number r such that

$$\int_0^\tau \|G(\cdot, s)\| \phi(s)\, ds \leqslant 1 \tag{2.2}$$

and $|\mathbf{F}[\mathbf{y}](t)| \leqslant \phi(t)\, r$ for $\|\mathbf{y} - \mathbf{h}\| \leqslant r$ and $t \in [0, \tau]$.

Now define the following sets:

$$Q_k = \{p: p \text{ is a polynomial of degree } k \text{ or less}\},$$

$$W_k = \{\mathbf{p}: \mathbf{p} = (p_1, p_2, ..., p_n)^{\mathrm{T}}, \text{ where } p_i \in Q_k \text{ for } 1 \leqslant i \leqslant n\},$$

$$P_k = \{\mathbf{p}: \mathbf{p} \in W_k \text{ and } M\mathbf{p}(0) + N\mathbf{p}(\tau) = \mathbf{b}\},$$

$$V_k = \{p: p \in Q_k \text{ and } |p(t)| \leqslant \phi(t)\, r \text{ for all } t \in [0, \tau]\},$$

$$U_k = \{\mathbf{p}: \mathbf{p} = (p_1, p_2, ..., p_n)^{\mathrm{T}}, \text{ where } p_i \in V_k \text{ for } 1 \leqslant i \leqslant n\},$$

and

$$S_k = \{\mathbf{p}: \mathbf{p} \in P_k \text{ and } \|\mathbf{p} - \mathbf{h}\| \leqslant r\}.$$

It should be noted that $\mathbf{h} \in S_k$ for all $k \geqslant n-1$.

We may now proceed to define our sequence of vector polynomials. Let $\mathbf{p} \in S_k$ for a fixed $k \geqslant n+1$. Since $\|\mathbf{p} - \mathbf{h}\| \leqslant r$, it follows that $|\mathbf{F}[\mathbf{p}](t)| \leqslant \phi(t)\, r$. Therefore $|F_i[\mathbf{p}](t)| \leqslant \phi(t)\, r$ for each $1 \leqslant i \leqslant n$, where $\mathbf{F}[\mathbf{p}] = (F_1[\mathbf{p}], F_2[\mathbf{p}], ..., F_n[\mathbf{p}])^{\mathrm{T}}$. Then using Theorems 3.1 and 4.1 from [9], for each $1 \leqslant i \leqslant n$, there exists a unique $v_i \in V_{k-n}$ such that

$$\inf_{v \in V_{k-n}} \|v - F_i[\mathbf{p}]\| = \|v_i - F_i[\mathbf{p}]\|. \tag{2.3}$$

Let $\mathbf{v}_0 = (v_1, v_2, ..., v_n)^{\mathrm{T}}$ and define the operator $B_k: S_k \to U_{k-n}$ by $B_k \mathbf{p} = \mathbf{v}_0$. Since $\mathbf{F}[\mathbf{y}]$ is uniformly continuous on compact sets, Theorem 4.4 of [9] implies that B_k is a continuous operator. For the same fixed $k \geqslant n+1$ let $\mathbf{v} \in U_{k-n}$ and set

$$\mathbf{q}(t) = \mathbf{h}(t) + \int_0^{\tau} G(t, s)\, \mathbf{v}(s)\, ds.$$

Define the operator H_k by $H_k \mathbf{v} = \mathbf{q}$. Using the properties of the Green's matrix it can be shown that H_k is a continuous operator from U_{k-n} into P_k. Finally, define the operator $T_k: S_k \to P_k$ by $T_k \mathbf{p} = H_k(B_k \mathbf{p})$. Since T_k is the composition of continuous operators, T_k is continuous. If $\mathbf{p}_k$ is a fixed point of T_k, i.e. $T_k \mathbf{p}_k = \mathbf{p}_k$, then

$$\mathbf{p}_k = \mathbf{h}(t) + \int_0^{\tau} G(t, s)\, \mathbf{v}_{k-n}(s)\, ds, \tag{2.4}$$

where $\mathbf{v}_{k-n} = (v_{k-n,1}, v_{k-n,2}, ..., v_{k-n,n})^{\mathrm{T}}$ with

$$\inf_{v \in V_{k-n}} \|v - F_i[\mathbf{p}_k]\| = \|v_{k-n,i} - F_i[\mathbf{p}_k]\|. \tag{2.5}$$

Therefore

$$\inf_{\mathbf{v} \in U_{k-n}} \|\mathbf{v} - F[p_k]\| = \|\mathbf{v}_{k-n} - \mathbf{F}[\mathbf{p}_k]\|$$

and from (2.4)

$$\inf_{\mathbf{v} \in U_{k-n}} \|\mathbf{v} - \mathbf{F}[\mathbf{p}_k]\| = \|\mathbf{p}_k' - E\mathbf{p}_k - \mathbf{F}[\mathbf{p}_k]\|. \tag{2.6}$$

Our approximating polynomials are then the fixed points of T_k for each $k \geqslant n+1$.

3. Existence of Fixed Points

The first step is to establish the existence of fixed points of T_k for each $k \geqslant n+1$.

THEOREM 1. *For fixed $k \geqslant n+1$ the operator T_k has a fixed point $\mathbf{p}_k$.*

Proof. To prove this theorem we will use Schauder's Fixed Point Theorem (see [6]). It is already known that T_k is a continuous map from a compact convex subset S_k of the Banach space W_k into $P_k \subset W_k$. Therefore we need only show that $T_k(S_k) \subset S_k$.

Let $\mathbf{p} \in S_k$. If $\mathbf{v}_0 = B_k \mathbf{p}$, where $\mathbf{v}_0 = (v_1, ..., v_n)^{\mathrm{T}}$, then $|v_i(t)| \leqslant \phi(t)\, r$ for all $t \in [0, \tau]$ and $1 \leqslant i \leqslant n$. Therefore $|\mathbf{v}_0(t)| \leqslant \phi(t)\, r$ for all $t \in [0, \tau]$. Define $\mathbf{q} = T_k \mathbf{p}$. Then

$$\mathbf{q}(t) = \mathbf{h}(t) + \int_0^\tau G(t, s)\, \mathbf{v}_0(s)\, ds.$$

This implies that $\mathbf{q} \in P_k$ and

$$\begin{aligned} |\mathbf{q}(t) - \mathbf{h}(t)| &\leqslant \int_0^\tau \|G(\cdot, s)\|\, |\mathbf{v}_0(s)|\, ds \\ &\leqslant r \int_0^\tau \|G(\cdot, s)\| \phi(s)\, ds \\ &\leqslant r \end{aligned}$$

for all $t \in [0, \tau]$. Therefore $\|\mathbf{q} - \mathbf{h}\| \leqslant r$, which completes our proof.

4. Convergence of Fixed Points

We should now determine a relationship between our sequence of fixed points of T_k and a solution of (1.1). We will prove, under certain conditions, that if $\mathbf{p}_k$ is a fixed point of T_k for each $k \geqslant n+1$ then $\mathbf{p}_k \to \mathbf{y}$ uniformly as $k \to \infty$, where $\mathbf{y}$ is a solution of (1.1). Also $\mathbf{p}'_k \to \mathbf{y}'$ uniformly on $[0, \tau]$. In order to establish these results we will need the following lemma:

LEMMA 1. *For each $k \geqslant n+1$ let $\mathbf{p}_k \in S_k$ be a fixed point of T_k. Let*

$$\mathbf{e}_k(t) = \mathbf{p}'_k(t) - E\mathbf{p}_k(t) - \mathbf{F}[\mathbf{p}_k](t), \qquad t \in [0, \tau]; \tag{4.1}$$

then $\lim_{k \to \infty} \|\mathbf{e}_k\| = 0$.

Proof. Let $\mathbf{v}_{k-n} = \mathbf{p}'_k - E\mathbf{p}_k$ for each $k \geqslant n+1$ and denote the components of $\mathbf{v}_{k-n}$ by $v_{i,k-n}$, where $1 \leqslant i \leqslant n$. Since $\mathbf{p}_k$ is a fixed point of T_k we have that

$$\inf_{v \in V_{k-n}} \|v - F_i[\mathbf{p}_k]\| = \|v_{i,k-n} - F_i[\mathbf{p}_k]\| \tag{4.2}$$

for each $1 \leqslant i \leqslant n$.

Further, for each $1 \leqslant i \leqslant n$, there exists $q_{i,k-n} \in Q_{k-n}$ such that

$$\inf_{v \in Q_{k-n}} \|v - F_i[\mathbf{p}_k]\| = \|q_{i,k-n} - F_i[\mathbf{p}_k]\|. \tag{4.3}$$

From Jackson's theorem (see [4]), if $\hat{e}_{i,k} = q_{i,k-n} - F_i[\mathbf{p}_k]$ then $\|\hat{e}_{i,k}\| \leqslant \omega_{i,k}(\alpha/(k-n))$, $1 \leqslant i \leqslant n$, $k \geqslant n+1$, for some constant α. independent of k, where $\omega_{i,k}$ is the modulus of continuity of $F_i[\mathbf{p}_k]$ for each i and k.

For each $k \geqslant n+1$ we have that $|\mathbf{v}_{k-n}(t)| \leqslant 2\|\phi\| r$ and $|\mathbf{p}_k(t)| \leqslant r + \|\mathbf{h}\|$ for all $t \in [0, \tau]$. Therefore $|\mathbf{p}'_k(t)| \leqslant \|E\| [r + \|\mathbf{h}\|] + 2\|\phi\| r$ for all $t \in [0, \tau]$. Then using the mean value theorem we can conclude that the sequence $\{\mathbf{p}_k\}_{k=n+1}^{\infty}$ is uniformly bounded and equicontinuous on $[0, \tau]$.

Given $\varepsilon > 0$ such that $\phi(t) r - \varepsilon > 0$ for all $t \in [0, \tau]$, set $\lambda = \varepsilon/[2(\varepsilon + r\|\phi\|)]$. Since each $F_i[\mathbf{y}](t)$ is uniformly continuous on compact sets and the sequence $\{\mathbf{p}_k\}$ is uniformly bounded and equicontinuous there exist positive numbers δ_i, $1 \leqslant i \leqslant n$, such that

$$|F_i[\mathbf{p}_k](t) - F_i[\mathbf{p}_k](s)| < [\lambda/(1-\lambda)]\, \varepsilon/2 \tag{4.4}$$

whenever $|t - s| < \delta_i$, $t, s \in [0, \tau]$, for each $k \geqslant n+1$. This implies that $\omega_{i,k}(\delta_i) \leqslant [\lambda/(1-\lambda)]\, \varepsilon/2$, $1 \leqslant i \leqslant n$. independent of k. Let $K_1 \geqslant n+1$ be large enough so that $\alpha/(k-n) \leqslant \min\{\delta_i\}$ for $k \geqslant K_1$. Therefore

$$\|\hat{e}_{i,k}\| \leqslant [\lambda/(1-\lambda)]\, \varepsilon/2 \tag{4.5}$$

for each $1 \leqslant i \leqslant n$ and $k \geqslant K_1$.

Define $g(t) = \phi(t) r - \varepsilon$ and for each $k \geqslant n+1$ let $s_{k-n}(t)$ be a polynomial of degree $k-n$ or less such that

$$\inf_{v \in Q_{k-n}} \|v - g\| = \|s_{k-n} - g\|. \tag{4.6}$$

There exists a number $K_2 \geqslant n+1$ such that $\|s_{k-n} - g\| < \varepsilon/2$ for all $k \geqslant K_2$. Let $K = \max\{K_1, K_2\}$.

For each $k \geqslant K$ and $1 \leqslant i \leqslant n$ define

$$b_{i,k-n}(t) = \lambda\, s_{k-n}(t) + (1-\lambda)\, q_{i,k-n}(t).$$

Then $b_{i,k-n} \in Q_{k-n}$ for each $k \geqslant K$ and $1 \leqslant i \leqslant n$ and

$$2\lambda(\phi(t) r - \varepsilon) - \phi(t) r < b_{i,k-n}(t) < \phi(t) r$$

for all $t \in [0, \tau]$. This implies that $|b_{i,k-n}(t)| < \phi(t)r$ for all $t \in [0, \tau]$, $1 \leqslant i \leqslant n$, and $k \geqslant K$. Therefore $b_{i,k-n} \in V_{k-n}$ for $1 \leqslant i \leqslant n$ and $k \geqslant K$.

We also have, for $t \in [0, \tau]$, $1 \leqslant i \leqslant n$, and $k \geqslant K$, that

$$\begin{aligned} &|b_{i,k-n}(t) - F_i[\mathbf{p}_k](t)| \\ &\quad \leqslant \lambda |s_{k-n}(t) - F_i[\mathbf{P}_k](t)| + (1-\lambda)|q_{i,k-n}(t) - F_i[\mathbf{p}_k](t)| \\ &\quad \leqslant \lambda\varepsilon/2 + \lambda[|\phi(t)r - F_i[\mathbf{p}_k](t)| + \varepsilon] + (1-\lambda)[\lambda/(1-\lambda)]\,\varepsilon/2 \\ &\quad \leqslant \lambda\varepsilon + \lambda[2r\|\phi\| + \varepsilon] = 2\lambda(\varepsilon + r\|\phi\|) = \varepsilon. \end{aligned}$$

Therefore, $\|b_{i,k-n} - F_i[\mathbf{p}_k]\| < \varepsilon$ for all $k \geqslant K$, $1 \leqslant i \leqslant n$. Since $b_{i,k-n} \in V_{k-n}$, $k \geqslant K$, $1 \leqslant i \leqslant n$, then

$$0 \leqslant \|v_{i,k-n} - F_i[\mathbf{p}_k]\| \leqslant \|b_{i,k-n} - F_i[\mathbf{p}_k]\| < \varepsilon$$

for each $1 \leqslant i \leqslant n$ and $k \geqslant K$. This implies that $\|\mathbf{v}_{k-n} - \mathbf{F}[\mathbf{p}_k]\| < \varepsilon$ for all $k \geqslant K$ and the proof is complete.

With Lemma 1 we can now establish the main results of this paper.

THEOREM 2. *If* $\mathbf{p}_k \in S_k$ *is a fixed point of* T_k *for each* $k \geqslant n+1$ *then there exists a function* $\mathbf{y}$, *whose components are in* $C^1[0, \tau]$, *and a subsequence* $\{\mathbf{p}_{k(j)}\}_{j=1}^{\infty}$ *of* $\{\mathbf{p}_k\}_{k=n+1}^{\infty}$ *such that* $\lim_{j\to\infty} \|(\mathbf{p}_{k(j)})^{(i)} - \mathbf{y}^{(i)}\| = 0$, $i = 0, 1$. *Moreover* $\mathbf{y}$ *is a solution to* (1.1).

Proof. In the proof of Lemma 1 it was established that the sequence $\{\mathbf{p}_k\}_{k=n+1}^{\infty}$ is equicontinuous and uniformly bounded on $[0, \tau]$. By Ascoli's theorem there is a subsequence $\{\mathbf{p}_{k(j)}\}_{j=1}^{\infty}$ such that $\mathbf{p}_{k(j)} \to \mathbf{y}$ uniformly on $[0, \tau]$ for some $\mathbf{y}$ whose components are in $C[0, \tau]$. If $\mathbf{e}_k = \mathbf{p}'_k - E\mathbf{p}_k - \mathbf{F}[\mathbf{p}_k]$ then from Lemma 1 $\mathbf{p}'_{k(j)} \to E\mathbf{y} + \mathbf{F}[\mathbf{y}]$ uniformly on $[0, \tau]$ as $j \to \infty$. Since $\mathbf{p}_{k(j)}$ is a fixed point of $T_{k(j)}$

$$\mathbf{p}_{k(j)}(t) = \mathbf{h}(t) + \int_0^{\tau} G(t, s)[\mathbf{p}'_{k(j)}(s) - E\mathbf{p}_{k(j)}(s)]\, ds$$

and thus $\mathbf{y} \in C^1[0, \tau]$ is a solution to (1.1). We then get, since $\mathbf{y}' = E\mathbf{y} + \mathbf{F}[\mathbf{y}]$, that $\mathbf{p}'_{k(k)} \to \mathbf{y}'$ uniformly on $[0, \tau]$ as $j \to \infty$ and our proof is complete.

5. RATE OF CONVERGENCE

We will now investigate the rate of convergence of a sequence of fixed points developed in Section 2. Here it will be assumed that we have a sequence of fixed points $\{\mathbf{p}_k\}$ and a solution, $\mathbf{y}$, of (1.1). Also, in addition to the conditions already placed on $\mathbf{f}$ we will assume that there exists a positive number K such that

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leqslant K\|\mathbf{x}_1 - \mathbf{x}_2\| \tag{5.1}$$

for all $t \in [0, \tau]$ and $\|\mathbf{h} - \mathbf{x}_i\| \leqslant r$, $i = 1, 2$. In addition let c be a number such that

$$\int_0^\tau \|G(\cdot, s)\| \, ds \leqslant c. \tag{5.2}$$

THEOREM 3. *If $Kc < 1$, then there is a constant β, independent of k, such that*

$$\|\mathbf{p}_k^{(i)} - \mathbf{y}^{(i)}\| \leqslant \beta\|\mathbf{e}_k\|, \qquad i = 0, 1,$$

where $\mathbf{e}_k = \mathbf{p}_k' - E\mathbf{p}_k - \mathbf{F}[\mathbf{p}_k]$.

Proof. Since $\mathbf{y}$ is a solution to (1.1) and $\mathbf{p}_k$ is a fixed point of T_k we have

$$\mathbf{y}(t) = \mathbf{h}(t) + \int_0^\tau G(t, s)\, \mathbf{F}[\mathbf{y}](s)\, ds$$

and

$$\mathbf{p}_k(t) = \mathbf{h}(t) + \int_0^\tau G(t, s)[\mathbf{p}_k'(s) - E\mathbf{p}_k(s)]\, ds$$

for each k. Here, recall, $\mathbf{F}[\mathbf{y}](t) = \mathbf{f}(t, \mathbf{y})$. Therefore

$$\|\mathbf{p}_k - \mathbf{y}\| \leqslant c\|\mathbf{e}_k\| + Kc\|\mathbf{p}_k - \mathbf{y}\|.$$

This implies that

$$\|\mathbf{p}_k - \mathbf{y}\| \leqslant [c/(1 - Kc)]\|\mathbf{e}_k\|. \tag{5.3}$$

We also have that

$$\begin{aligned} \|\mathbf{p}_k' - \mathbf{y}'\| &\leqslant \|\mathbf{e}_k\| + [\|E\| + K]\|\mathbf{p}_k - \mathbf{y}\| \\ &\leqslant [(1 + c\|E\|)/(1 - Kc)]\|\mathbf{e}_k\|. \end{aligned} \tag{5.4}$$

This, then, implies our result.

Two consequences of Theorem 3 should be discussed. First, combining Lemma 1, Theorem 2, and Theorem 3 we get the existence of a solution to (1.1) and the convergence of the entire sequence $\{\mathbf{p}_k\}$ of fixed points to that solution. Second, it can be easily shown that the conditions of Theorem 3 guarantee that (1.1) has a unique solution.

6. Computation of Fixed Points

We now turn to the task of computing a fixed point of T_k. Let $\mathbf{p}_0$ be any element of S_k for a fixed $k \geqslant n+1$ (this may be taken to be $\mathbf{h}$). Define $\mathbf{p}_{m+1} = T_k \mathbf{p}_m$ for $m = 0, 1, 2, \ldots$. We know that $\mathbf{p}_m \in S_k$ for each $m \geqslant 0$. Since S_k is compact the sequence $\{\mathbf{p}_m\}$ has a cluster point $\mathbf{p} \in S_k$. Therefore, there exists a subsequence $\{\mathbf{p}_{m(j)}\}_{j=1}^{\infty}$ such that $\mathbf{p}_{m(j)} \to \mathbf{p}$ uniformly on $[0, \tau]$ as $j \to \infty$. We now proceed to show that $\mathbf{p}$ is a fixed point of T_k.

Let $\mathbf{v}_m = \mathbf{p}'_{m+1} - E\mathbf{p}_{m+1}$ for each $m \geqslant 0$. Then $\mathbf{v}_m = (\mathbf{v}_{1,m}, \ldots, v_{n,m})^{\mathrm{T}}$, where

$$\inf_{v \in V_{k-n}} \|v - F_i[\mathbf{p}_m]\| = \|v_{i,m} - F_i[\mathbf{p}_m]\| \tag{6.1}$$

for $m \geqslant 0$, $1 \leqslant i \leqslant n$. It will be assumed that, for each i, $F_i[\mathbf{p}]$ is not contained in Q_k when $\mathbf{p}$ is contained in W_k. This will imply that $\|\mathbf{v}_{i,m} - F_i[\mathbf{p}_m]\| \neq 0$ for $m \geqslant 0$, $1 \leqslant i \leqslant n$. Define the following sets for each $m \geqslant 0$ and $1 \leqslant i \leqslant n$:

$$\begin{aligned}
Y^{i,m}_{+1} &= \{t \in [0, \tau] \colon F_i[\mathbf{p}_m](t) - v_{i,m}(t) = \|F_i[\mathbf{p}_m] - v_{i,m}\|\},\\
Y^{i,m}_{-1} &= \{t \in [0, \tau] \colon F_i[\mathbf{p}_m](t) - v_{i,m}(t) = -\|F_i[\mathbf{p}_m] - v_{i,m}\|\},\\
Y^{i,m}_{+2} &= \{t \in [0, \tau] \colon v_{i,m}(t) = -\phi(t) r\},\\
Y^{i,m}_{-2} &= \{t \in [0, \tau] \colon v_{i,m}(t) = \phi(t) r\}
\end{aligned}$$

and

$$Y^{i,m}_p = Y^{i,m}_{+1} \cup Y^{i,m}_{-1} \cup Y^{i,m}_{+2} \cup Y^{i,m}_{-2}.$$

Since $|F_i[\mathbf{p}_m](t)| \leqslant \phi(t) r$ for all $t \in [0, \tau]$, $1 \leqslant i \leqslant n$, and $m \geqslant 0$, we have that

$$(Y^{i,m}_{+1} \cup Y^{i,m}_{+2}) \cap (Y^{i,m}_{-1} \cup Y^{i,m}_{-2}) = \varnothing \tag{6.2}$$

for each $1 \leqslant i \leqslant n$ and $m \geqslant 0$. Then define $\sigma_{i,m}(t) = -1$ if $t \in Y^{i,m}_{-1} \cup Y^{i,m}_{-2}$ and $\sigma_{i,m}(t) = +1$ if $t \in Y^{i,m}_{+1} \cup Y^{i,m}_{+2}$ for each $1 \leqslant i \leqslant n$ and $m \geqslant 0$.

From Theorem 3.2 of [9], there exist $k-n+2$ consecutive points $t_{1,i,m} < t_{2,i,m} < \cdots < t_{k-n+2,i,m}$ in $Y^{i,m}_p$ satisfying

$$\sigma_{i,m}(t_{l,i,m}) = (-1)^{l+1} \sigma_{i,m}(t_{1,i,m}) \tag{6.3}$$

for each $1 \leqslant i \leqslant n$, $m \geqslant 0$, and $1 \leqslant l \leqslant k-n+2$. For each $1 \leqslant i \leqslant n$ and $m \geqslant 0$ let $X_{i,m} = \{t_{1,i,m}, \ldots, t_{k-n+2,i,m}\}$. The sequence $\{X_{i,m}\}_{m=0}^{\infty}$ is contained in the compact set $[0, \tau]^{k-n+2}$ for each $1 \leqslant i \leqslant n$. Therefore they have cluster points $X_i = \{t_{1,i}, \ldots, t_{k-n+2,i}\}$, $1 \leqslant i \leqslant n$. Without loss of generality, assume that all subsequences from $\{\mathbf{p}_m\}$ and $\{X_{i,m}\}$ that converge to $\mathbf{p}$ and X_i,

$1 \leqslant i \leqslant n$, involve the same indices. These subsequences will be denoted by $\{\mathbf{p}_{m(j)}\}_{j=1}^{\infty}$ and $\{X_{i,m(j)}\}_{j=1}^{\infty}$, $1 \leqslant i \leqslant n$.

Let $\mathbf{e}_m(t) = \mathbf{v}_m(t) - \mathbf{F}[\mathbf{p}_m](t) = \mathbf{p}'_{m+1}(t) - E\mathbf{p}_{m+1}(t) - \mathbf{F}[\mathbf{p}_m](t)$ for each $m \geqslant 0$. Denote the components of $\mathbf{e}_m$ by $e_{i,m}$, $1 \leqslant i \leqslant n$. Also define $\mathbf{u}(t) = \mathbf{p}'(t) - E\mathbf{p}(t)$ and $\mathbf{e}(t) = \mathbf{u}(t) - \mathbf{F}[\mathbf{p}](t)$ with the components of $\mathbf{e}$ given by e_i, $1 \leqslant i \leqslant n$.

THEOREM 4. *If for each $t_{l,i}$, $t_{l+1,i} \in X_i$, $1 \leqslant i \leqslant n$, $1 \leqslant l \leqslant k-n+1$, we have*

$$\operatorname{sgn}[e_i(t_{l,i})] = -\operatorname{sgn}[e_i(t_{l+1,i})], \tag{6.4}$$

then $\mathbf{p}$ is a fixed point of T_k.

Proof. Define $\mathbf{q} = T_k\mathbf{p}$, $\mathbf{v} = \mathbf{q}' - E\mathbf{q}$ and $\mathbf{d} = \mathbf{v} - \mathbf{F}[\mathbf{p}]$ with the components of $\mathbf{d}$ denoted by d_i, $1 \leqslant i \leqslant n$. Also for each $1 \leqslant i \leqslant n$ define the following sets:

$$Y^i_{+1} = \{t \in [0, \tau] : F_i[\mathbf{p}](t) - v_i(t) = \|F_i[\mathbf{p}] - v_i\|\},$$
$$Y^i_{-1} = \{t \in [0, \tau] : F_i[\mathbf{p}](t) - v_i(t) = -\|F_i[\mathbf{p}] - v_i\|\},$$
$$Y^i_{+2} = \{t \in [0, \tau] : v_i(t) = -\phi(t)\, r\},$$
$$Y^i_{-2} = \{t \in [0, \tau] : v_i(t) = \phi(t)\, r\}$$

and

$$Y_p = Y^i_{+1} \cup Y^i_{-1} \cup Y^i_{+2} \cup Y^i_{-2}.$$

Again, we have that

$$(Y^i_{+1} \cup Y^i_{+2}) \cap (Y^i_{-1} \cup Y^i_{-2}) = \varnothing \tag{6.5}$$

for each $1 \leqslant i \leqslant n$. Then define $\sigma_i(t) = -1$ if $t \in Y^i_{-1} \cup Y^i_{-2}$ and $\sigma_i(t) = +1$ if $t \in Y^i_{+1} \cup Y^i_{+2}$, $1 \leqslant i \leqslant n$. Using Theorem 4.4 of [9] and the fact that the family $\{\mathbf{F}[\mathbf{p}_m]\}$ is equicontinuous on $[0, \tau]$ we have that $e_{i,m(j)}(t_{l,i,m(j)}) \to d_i(t_{l,i})$ as $j \to \infty$, for each $1 \leqslant l \leqslant k-n+2$ and $1 \leqslant i \leqslant n$. Also, $\|e_{i,m(j)}\| \to \|d_i\|$ for each $1 \leqslant i \leqslant n$, as $j \to \infty$. This gives us the following relationships as $j \to \infty$: $Y^{i,m(j)}_{+1} \to Y^i_{+1}$, $Y^{i,m(j)}_{-1} \to Y^i_{-1}$, $Y^{i,m(j)}_{+2} \to Y^i_{+2}$, $Y^{i,m(j)}_{-2} \to Y^i_{-2}$ and $Y^{i,m(j)}_p \to Y^i_p$ for each $1 \leqslant i \leqslant n$. Therefore

$$\sigma_i(t_{l,i}) = (-1)^{l+1}\sigma_i(t_{1,i}), \qquad 1 \leqslant l \leqslant k-n+2, \tag{6.6}$$

for each $1 \leqslant i \leqslant n$.

We have that for each $1 \leqslant i \leqslant n$ and $1 \leqslant l \leqslant k-n+2$,

$$u_i(t_{l,i}) - v_i(t_{l,i}) = d_i(t_{l,i}) - e_i(t_{l,i}). \tag{6.7}$$

Also, using (6.4) and Theorem 4.2 of [9] we have

$$\|F_i[\mathbf{p}]-v_i\| \geqslant |F_i[\mathbf{p}](t_{l,i})-u_i(t_{l,i})|, \qquad 1\leqslant l\leqslant k-n+2, \tag{6.8}$$

for each $1\leqslant i\leqslant n$. Then using (6.7) and (6.8) we can conclude that $u_i(t_{l,i})-v_i(t_{l,i})\geqslant 0$ if $t_{l,i}\in Y^i_{+1}\cup Y^i_{+2}$ and $u_i(t_{l,i})-v_i(t_{l,i})\leqslant 0$ if $t_{l,i}\in Y^i_{-1}\cup Y^i_{-2}$, $1\leqslant l\leqslant k-n+2$, $1\leqslant i\leqslant n$. Therefore, from (6.6), it follows that for each $1\leqslant i\leqslant n$, $u_i(t)-v_i(t)$ has $k-n+1$ zeros in $[0,\tau]$. Since u_i and v_i are polynomials of degree $k-n$ or less for each $1\leqslant i\leqslant n$, then $u_i(t)\equiv v_i(t)$ for each $1\leqslant i\leqslant n$. Thus $\mathbf{u}(t)\equiv\mathbf{v}(t)$, which completes the proof.

It should be noted that if the sequence $\{\mathbf{p}_m\}_{m=0}^{\infty}$ is such that $\mathbf{p}_m\to\mathbf{p}$ uniformly as $m\to\infty$ then (6.4) is always satisfied.

7. Scalar Equations and the Best Approximation

Consider the system

$$\begin{aligned} &y^{(n)}(t)=f(t,y,\dots,y^{(n-1)}), && t\in[0,\tau],\\ &\sum_{j=1}^{n} c_{ij}y^{(j-1)}(0)+d_{ij}y^{(j-1)}(\tau)=b_i, && 1\leqslant i\leqslant n,\end{aligned} \tag{7.1}$$

where f is a continuous real valued scalar function on $[0,\tau]\times R^n$ and c_{ij}, d_{ij}, and b_i are real constants for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$. Using standard techniques this system may be transformed to the form (1.1), where $\mathbf{y}=(y,y',\dots,y^{(n-1)})^{\mathrm{T}}$. Therefore all of the theory of the preceding sections applies to (7.1) transformed to (1.1). However, additional things may be said about (7.1).

First, let $E_k=\{p{:}\, p\in Q_k \text{ and } \sum_{j=1}^{n} c_{ij}p_{(0)}^{(j-1)}+d_{ij}p_{(\tau)}^{(j-1)}=b_i,\ 1\leqslant i\leqslant n\}$. Then the best approximation from E_k, for a given k, is a polynomial q_k satisfying

$$\inf_{q\in E_k}\|q^{(n)}-f(\cdot,q,\dots,q^{(n-1)})\|=\|q_k^{(n)}-f(\cdot,q_k,\dots,q_k^{(n-1)})\|. \tag{7.2}$$

If we modify all of our sets in Section 2 by replacing the vectors $(p_1,p_2,\dots,p_n)^{\mathrm{T}}$ with $(p,p',\dots,p^{(n-1)})^{\mathrm{T}}$ and name the sets $\hat{W}_k$, $\hat{P}_k$, $\hat{U}_k$, and $\hat{S}_k$, then finding a polynomial q_k which satisfies (7.2) is equivalent to finding a vector polynomial $\mathbf{q}_k=(q_k,q_k',\dots,q_k^{(n-1)})^{\mathrm{T}}$ which satisfies

$$\inf_{\mathbf{q}\in\hat{P}_k}\|\mathbf{q}'-E\mathbf{q}-\mathbf{F}[\mathbf{q}]\|=\|\mathbf{q}_k'-E\mathbf{q}_k-\mathbf{F}[\mathbf{q}_k]\|, \tag{7.3}$$

where $\mathbf{F}[\mathbf{y}](t)=\mathbf{f}(t,\mathbf{y})$. Then in light of Theorem 4.3 of [9] we can apply the same type of analysis as that applied in [3] to find a vector polynomial $\mathbf{q}_k$ satisfying (7.3). (We would use the fact that if $\mathbf{p}=(p,p',\dots,p^{(n-1)})^{\mathrm{T}}$ then $\|\mathbf{p}'-E\mathbf{p}-\mathbf{F}[\mathbf{p}]\|=\|p^{(n)}-f(\cdot,p,\dots,p^{(n-1)})\|$.)

In converting system (7.1) into (1.1) and checking the condition given in (2.2), we may be losing some examples. Using the method in [3] for scalars and that of this paper we can include scalar equations that would have been eliminated in the conversion process. Thus condition (2.2) would become

$$\int_0^\tau \|H(\cdot, s)\| \phi(s)\, ds \leqslant 1 \tag{7.4}$$

and

$$\int_0^\tau \|H_t(\cdot, s)\| \phi(s)\, ds \leqslant 1, \tag{7.5}$$

etc., where $H(t, s)$ is the Green's function for the problem

$$\begin{aligned} &y^{(n)}(t) = 0 \\ &\sum_{j=1}^{n} c_{ij} y^{(j-1)}(0) + d_{ij} y^{(j-1)}(\tau) = 0, \qquad 1 \leqslant i \leqslant n. \end{aligned} \tag{7.6}$$

This can be found by computing the Green's matrix $G(t, s)$ and then letting $H(t, s) = \mathbf{d}_1^T G(t, s)\, \mathbf{d}_n$, where $\mathbf{d}_1 = (1, 0, 0, ..., 0)^{\mathrm{T}}$ and $\mathbf{d}_n = (0, 0, ..., 0, 1)^{\mathrm{T}}$.

Finally, if we simplify the boundary conditions in (7.1) to $y(0) = b_1$, $y(\tau) = b_2$, there are many other theorems we can prove using the technique of this paper. One simple problem is as follows: Consider

$$\begin{aligned} y''(t) &= f(t, y), \qquad t \in [0, \tau], \\ y(0) &= b_1, y(\tau) = b_2. \end{aligned} \tag{7.7}$$

Here, our Green's function $H(t, s)$ is nonpositive for all $s, t \in [0, \tau]$. If we let $h(t) = [(b_2 - b_1)/\tau]\, t + b_1$ then our condition would be that there is a positive function $\phi(t)$ on $[0, \tau]$ such that

$$-\int_0^\tau H(t, s)\, \phi(s)\, ds \leqslant 1 \tag{7.8}$$

for all $t \in [0, \tau]$. Also, there exists an $r > 0$ such that $0 \leqslant f(t, y) \leqslant \phi(t)\, r$ when $\|y - h\| \leqslant r$ and $y(t) \leqslant h(t)$ for all $t \in [0, \tau]$. With these conditions we can follow the procedure of this paper and produce the theorems needed.

8. Examples

In this section four examples are presented. In each case the algorithm introduced in Section 6 was used to produce a fixed point. Each of the four examples is a scalar system of the form (7.1) with $f(t, y, \ldots, y^{(n-1)}) = a(t)\, y^m$, where m is a positive real number and $a(t) \neq 0$ for all $t \in [0, \tau]$. Also, we will assume $\tau = 1$. In this case the corresponding scalar conditions, described in Section 7, which will insure the existence of a fixed point become: There exists a function $\phi \in \subset [0, 1]$, with $\phi(t) > 0$ for all $t \in [0, 1]$, and a real number r such that

$$\int_0^1 \|H(\cdot, s)\| \phi(s)\, ds \leqslant 1$$

and $|a(t)[y(t)]^m| \leqslant \phi(t)\, r$ for $\|y - h\| \leqslant r$, $t \in [0, 1]$. Here, $H(t, s)$ is the Green's function described in Section 7. If we let

$$A = \frac{1}{\int_0^1 \|H(\cdot, s)\|\, |a(s)|\, ds}$$

and $\phi(t) = A|a(t)|$ then a condition which would insure the existence of a fixed point is given by: There exists a real number r such that

$$\frac{[r + \|h\|]^m}{r} \leqslant A. \tag{8.1}$$

This condition can be checked numerically on the computor.

In order to calculate these fixed points it was necessary to calculate a best restricted approximation and thus the algorithm developed by G. D. Taylor and M. J. Winter [10] was used for this purpose.

EXAMPLE 1.

$$y''' + 6y^4 = 0, \qquad t \in [0, 1],$$
$$y(0) = \tfrac{1}{2}, \qquad y'(0) = -\tfrac{1}{4}, \qquad y(1) = \tfrac{1}{3}.$$

Here, $\|H(\cdot, s)\| = s(1-s)^2/2(2-s)$, where $s \in [0, 1]$. Thus $A = 1/(-4 + 6 \ln 2)$ and $\|h\| = \frac{1}{2}$. Therefore, we can take $r = 0.0109$ in order to satisfy (8.1). Also, $K = 3(2r+1)^3$ and $C = -\frac{2}{3} + \ln 2$. This gives us that $KC < 0.085 < 1$, which implies that the conditions of Theorem 3 are satisfied (making the appropriate changes for scalar equations). A fixed point of T_7 is

$$\begin{aligned} P_7(t) = 0.5 - 0.25t + 0.12532746t^2 - 0.06249514t^3 \\ + 0.03114968t^4 - 0.01510905t^5 + 0.00648877t^6 \\ - 0.00202841t^7. \end{aligned}$$

A fixed point of T_8 is

$$P_8(t) = 0.5 - 0.25t + 0.12486999t^2 - 0.06249929t^3 + 0.03123048t^4 - 0.01547963t^5 + 0.00733774t^6 - 0.00299848t^7 + 0.00087248t^8.$$

The actual solution is $y(t) = 1/(t+2)$ and the uniform errors are

$$\|P_7 - y\| = 1.5 \times 10^{-4},$$
$$\|P_7' - y'\| = 1.9 \times 10^{-3},$$
$$\|P_7'' - y''\| = 1.7 \times 10^{-2},$$
$$\|P_7''' - y'''\| = 1.1 \times 10^{-1},$$
$$\|P_8 - y\| = 6.5 \times 10^{-5},$$
$$\|P_8' - y'\| = 8.9 \times 10^{-4},$$
$$\|P_8'' - y''\| = 8.9 \times 10^{-3},$$

and

$$\|P_8''' - y'''\| = 6.4 \times 10^{-2}.$$

EXAMPLE 2.

$$y'' = \frac{2}{t+1} y^2, \qquad t \in [0, 1],$$
$$y(0) = 1, \qquad y(1) = \tfrac{1}{2}.$$

In this case $\|H(\cdot, s)\| = s(1-s)$, where $s \in [0, 1]$, and $\|h\| = 1$. Therefore $A = 1/(3 - 4 \ln 2)$ and thus any r between 0.537763 and 1.859557 will satisfy (8.1). However, $K = 4(1+r)$ and $C = \frac{1}{6}$, which implies that $KC = \frac{2}{3}(1+r) > 1.025$. A fixed point of T_7 is given by

$$P_7(t) = 1 - 1.00000733t + 0.99947011t^2 - 0.98460124t^3 + 0.89011394t^4 - 0.63302537t^5 + 0.28677896t^6 - 0.05872908t^7.$$

The actual solution is $y(t) = 1/(t+1)$ and the uniform errors are

$$\|P_7 - y\| = 8.2 \times 10^{-6},$$
$$\|P_7' - y'\| = 9.0 \times 10^{-5},$$

and

$$\|P_7'' - y''\| = 1.1 \times 10^{-3}.$$

The last two examples do not satisfy the conditions of the main body of the paper but are examples of the form (7.7). They do satisfy (7.8) and the conditions which follow (7.8).

EXAMPLE 3.

$$y'' = e^{-t}y^2, \qquad t \in [0, 1],$$
$$y(0) = 1, y(1) = e = 2.7182818.$$

A fixed point of the appropriate operator, of degree 7, is

$$\begin{aligned} P_7(t) = 1 &+ 0.99999995t + 0.49999945t^2 + 0.16668016t^3 \\ &+ 0.04159046t^4 + 0.00852119t^5 + 0.00115943t^6 \\ &+ 0.00033116t^7. \end{aligned}$$

The actual solution is $y(t) = e^t$. The uniform errors are

$$\|P_7 - y\| = 4.0 \times 10^{-8},$$
$$\|P_7' - y'\| = 1.1 \times 10^{-7},$$

and

$$\|P_7'' - y''\| = 1.1 \times 10^{-6}.$$

EXAMPLE 4.

$$y'' = \tfrac{3}{2} y^2, \qquad t \in [0, 1],$$
$$y(0) = 4, \qquad y(1) = 1.$$

A fixed point of the operator of degree 7 is

$$\begin{aligned} P_7(t) = 4 &- 8.00028350t + 11.98571014t^2 - 15.58006579t^3 \\ &+ 16.94631392t^4 - 13.48051808t^5 + 6.51470248t^6 \\ &- 1.38585917t^7. \end{aligned}$$

The actual solution is $y(t) = 4/(t+1)^2$. The uniform errors are

$$\|P_7 - y\| = 2.0 \times 10^{-4},$$
$$\|P_7' - y'\| = 2.4 \times 10^{-3},$$

and

$$\|P_7'' - y''\| = 2.9 \times 10^{-2}.$$

Finally, it should be mentioned that none of the above examples satisfy the conditions imposed in [3] while all of the examples in [3] satisfy the conditions of this paper.

9. Conclusions

The objective of this paper was to generalize the results of [3]. This was done in two ways. First, the second order equation with boundary conditions was expanded to a system of equations with boundary conditions, which includes single higher order equations. Second, even for second order equations, the conditions under which the theorems hold have been relaxed with the help of best restricted range approximations [9]. In both cases the conditions given by (2.2) and below are needed to insure the existence of a fixed point. These conditions are more general than those given in [3] and thus a second order nonlinear boundary value problem satisfying the conditions set forth in [3] will satisfy the corresponding scalar conditions of (2.2) and below. It is also evident that many other theorems, not covered in this paper, can be generated by the technique used in composing two operators and then finding a fixed point of the resultant operator.

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